Love, Death, and Taxes: Applications of Real Options in Economic Systems

Introduction

Modern economic systems have become increasingly complex as changes in the environment continue at warp speed. As complexity increases, risk also increases. Various stakeholders in these economic systems have applied vast amounts of intellectual and financial capital to develop mechanisms to describe and discuss complexity, harvest rewards/value from it, and manage the attendant risk. The critical platform for such endeavors, however, has remained virtually the same for over 500 years. It is a humdrum discipline called financial accounting.

Basic financial accounting is structured on a set of generally useful rules by which all agents within economic systems can describe themselves, their net resources, and their activities in a common language. This common language enables transactions to take place at all levels in the markets on a reasonably level playing field. It provides for the exchange of more or less complete information between parties. It provides the base on which strategic planning and competitive games can be designed and implemented. While the taxing and other regulatory authorities have introduced high levels of complexity into basic accounting systems, it is accounting systems that allow them to set their "handicaps" (game restrictions) and fund their activities. Current financial accounting rules are complicated but workable because, in the end, every agent resource and activity can be traced back to cash, the fundamental unit and common denominator of real-world exchange.

Although financial accounting serves a critical purpose that cannot be replaced, dynamic changes in the environment and the economic systems that populate it have created communications issues regarding the effects of such changes on firms. Since financial accounting is the common language for business resources and activities, it has gotten routinely criticized for not measuring up to the challenges in the environment, for not being able to represent the intangible and the esoteric in a universally accepted way. While no one has asked if any other discipline could do better, accounting regulators worldwide have made the decision to opt for "relevance" over "reliability" and invented a new discipline, Fair Value accounting, codified in November 2007 in Statement of Financial Accounting Standards No. 157, "Fair Value Measurement."

To those not interested or embroiled in accounting or valuation, it seems an innocuous move. Many in the finance community have welcomed it as an improvement over the dinosaur of historical cost basis accounting. Whether we know or care about SFAS 157, it is about to change all worlds. In effect, it has altered the base language of economic systems radically, in favor of constructed markets and hypothetical events built on valuation principals, rather than real-world ones based in actual transactions. While for years valuation professionals and in-house corporate finance teams have grappled with how to assign values to specific firm resources and projects without ignoring the other resources and projects that contribute to such values, this problem was primarily confined to the context of business acquisitions and certain types of business entities. The new Fair Value standard subjects the entire balance sheet (and income statement) of all firms to such treatment on an ongoing basis, effectively disaggregating the firm.

Since the contents of financial reports flow back into the markets as information on which real-world decisions are made at every level, fair value reporting will create significant and unforeseen effects in the "state of the system." New approaches to collecting and analyzing market and firm data will be required to manage the introduction of increased levels of subjectivity into these data.

This conceptual paper discusses a number of the challenges generated by economic system complexity and several of the solutions being offered. It demonstrates that the firm is an economic system in its own right and can be analyzed as such. It presents an approach to firm organizational design and valuation, using applications of real options “in” economic systems, and describes its relevance to the problems created by SFAS 157. It concludes with a presentation of avenues for future research.

I. Problem Statement

Modern economic systems are experiencing exponential growth in complexity, interdependence, and risk. The events creating such dramatic changes in the environment have been well discussed and analyzed by a broad range of stakeholders. Many of these are actively searching for improved methods of dealing with them. But non-proprietary, effective solutions are few and far between. The Financial Accounting Standards Board and its international counterpart, the International Accounting Standards Board, have recently added to the complexity by moving financial accounting away from its historical transaction basis (i.e., cost and cash) toward a prospective valuation basis (i.e., estimates and opinion). Assuming that this direction cannot be altered, the valuation profession will have to develop better tools to accomplish the mandates set before it. Real options analysis may offer such tools.

There is a nested set of related problems arising from this central one. A number of the key ones are stated following.

Dynamic changes in the environment are affecting the use of financial accounting, the foundational language and tool of modern economic systems.

The following are an abbreviated list of changes that have caused so much frustration to be directed at financial accounting.

Change 1: The "discovery" and influence of intangible assets on organizational structure, growth, complexity and value. As market attention has shifted toward intangible assets and away from tangible ones as the source of firm value, financial accounting faces a dilemma. It has to devise methods of measuring and recording the influence of such assets on the organization and its benefit streams (i.e. outputs). Yet, most of these assets are neither separable nor transferable, two attributes necessary for resource measurement and exchange.

Change 2: The explosion of operating and financial complexity in economic systems worldwide. Industry and cross-industry consolidation, globalization, and new and exotic markets, industries and products requires an increasingly broad range of organizational and transaction structures, many of which are highly complex. Financial accounting has been asked to address such complexity in meaningful and accurate ways.

Change 3: The increased presence of governmental and regulatory influence over every area of life. Organizations are expending massive amounts of time and resources to adapt to and mitigate the requirements of government and regulatory bodies. Tax rules continue to burgeon. Financial accounting must keep pace with the constant turbulence.

Change 4: The exponential increase in market and transaction complexity. Increasingly sophisticated investment vehicles, enhanced computer-based trading and desk-top trading, 24x7 markets, global currency flows, Internet collaboration, consolidating exchanges and exchanges that operate as public companies – these all feed on and generate financial information on a real-time basis. Financial accounting is being asked to incorporate high levels of complexity and speed into procedures and processes that were designed for lower levels of both.

Change 5: The increased potential of unforeseen and/or unforeseeable random acts of violence that disrupt economic systems on a global scale. Dr. Nassim Taleb calls these "black swans" and suggests that there is ". . . an ingrained tendency in humans to underestimate outliers . . . Left to our own devices, we tend to think that what happens every decade in fact only happens once every century, and, furthermore, that we know what's going on." (Taleb, 2007:141).

Increased complexity and turbulence create increased, and different, sources of risk, as well as new sets of questions.

Can modern economic systems afford to view risk and complexity in the traditional way? Should financial reporting more explicitly reflect the complexity and risk inherent in organizational resources and activities? Do we need better methods of describing both the state of the system and the economic systems functioning within it? Do modern economic systems actually function in the ways that are commonly described in the traditional literature and financial reporting? If they do not, how should we describe them? These are just a few of the issues that arise from the scale and scope of change within the global economic environment.

Financial valuation, yesterday's red-headed stepchild, has become today's darling as we attempt to resolve such issues. But is she ready?

Financial valuation, an esoteric hybrid of all business disciplines, is currently practiced in a linear, deterministic, but prospective, manner with varying degrees of rigor. There are many theoretical and practical issues on which no two valuation analysts fully agree. For this field, governed by informed professional judgment, beauty is truly in the eyes of the beholder. SFAS

157 institutes financial valuation as the arbiter of value for the net resources, and related benefit streams, of the firm for financial reporting purposes. Yet financial valuation has its own set of disabilities that have not been resolved.

Example 1: Traditional valuation approaches no longer suffice to capture the complex realities of dynamic economic systems. These same approaches will also be inadequate to meet the demands of fair value accounting.

Example 2: Non-linear valuation approaches, such as real options analysis, contain complexities and challenges that are beyond the average practitioner (e.g., difficulties in developing fundamental inputs for the models; difficulties of creating consistent structure and repeatability of RO problems and solutions; a low level of computational transparency for average users; no standardized methods for checking projected results against actual ones). What makes the post-SFAS 157 world different is that, although these familiar challenges will continue to inhere in real options analysis, the same challenges will now inhere in all aspects of accounting, a non-finance discipline. This, in turn, will ensure that a high degree of subjectivity and complexity will become embedded in both financial reporting and market prices.

The remainder of this paper will explore the potential contribution of real options analysis to resolving the stated problems.

II. Review of the Relevant Literature

To gain a fresh perspective, the fundamental research for this paper takes an eclectic approach, drawing on academic research, academic books, practitioner manuals and papers, Financial Accounting Standards Board pronouncements, and the popular press in a variety of disciplines such as valuation, real options analysis and risk management, physics and complexity science, and insider tips from "The Street." The focus was to:

- Explore the manner in which various agents within dynamic global economic systems describe, structure, and leverage increasingly high degrees of complexity and risk in the system;
- Identify what appear to be the most powerful and genuinely useful approaches to provide rigorous solutions to the stated problems.

Overview of Findings

At the top level, research findings were widely varied. The common theme was, "How do we deal with and create/harvest value from increasingly complex and risky systems?" Proposed solutions seemed to pursue three directions: 1) build complex models to describe and manage increasing complexity; 2) build simplified models to describe and manage increasing complexity; 3) build layered models that reduce complexity as they progress through a sequence. All of these approaches involve varying degrees of computational complexity and potential/actual "black boxes," i.e. lack of transparency.

The following are examples of approaches based on building complex models to describe and manage complex systems. They are both taken from the field of valuation and are written by practitioners, thus are not part of a body of academic literature.

Complex Models to Describe and Manage Complex Economic Systems

Model 1: Valuation of complex capital structures: "Th[e] growing trend toward fair value presents significant challenges in valuing privately-held companies with complex capital structures. Because it is necessary to first value those securities with superior claims to common equity, many valuation specialists, auditors, and financial executives now find themselves forced to enter a jungle of complex capital valuation. Depending upon the provisions associated with the components of a complex capital structure, accurate valuation of common stock in this environment may require sophisticated simulation models. Until recently, however, there was very little guidance – much less convergence of thought . . . within the appraisal community. Even where such guidance exists, it is unnecessarily conflicting, and more important, incapable of handling such commonplace features as cash distributions prior to liquidity events and performance-based vesting." [Chamberlain et al, 2007: 1]

To address this challenge, a team of valuation professionals and academics propose a methodology, based in simulation techniques, to integrate two extant valuation methods, the Options Pricing Method (OPM) and the Probability Weighted Expected Return Method (PWERM). These two methods are commonly used in the field, having been propounded in a 2004 AICPA Practice Aid, "Valuation of Privately-Held-Company Equity Securities Issued as Compensation."

"OPM takes as a starting point the current enterprise value and, using a volatility estimate that captures the market risk of the underlying business, models the stochastic evolution of this value over time. The various equity classes are then viewed as [European] option-like claims on this underlying value . . ." [Chamberlain et al, 2007: 4] The OPM is performed using the closed-form Black-Scholes option pricing method which does not allow for performance-based vesting or other path-dependent events.

"PWERM explicitly takes into account the random nature and timing of potential future liquidity events. . . . [C]urrent enterprise value is the probability-weighted sum of the discounted future liquidity outcomes." [Chamberlain et al, 2007: 4-5] The discount rate utilized is that of the underlying asset, thus preventing the method from capturing the changes in risk over time and over equity classes.

The integrative method incorporates the following steps: 1) Determining the risk-neutral distribution of underlying asset values; 2) simulate future asset values using this distribution and the selected end-points that represent various liquidity events; 3) infer benefit stream paths (EBITDA, cash flow) from these asset value paths; 4) use the benefit stream paths to determine the cash distributions resulting from path-dependent events (such as performance-based vesting) and the effect of such distributions on end-point liquidation values; 5) using traditional priority rules, allocate the enterprise values determined in Steps 1-4 to the various equity classes; 6) discount the resulting payoff values by the risk-free rate (since the underlying asset has been simulated under risk-neutral conditions); 7) repeat these steps and take an average to conclude a final value for each equity class. [Chamberlain et al, 2007: 8]

This model allows the analyst to set "conditions believed to resemble the ones that prevail in reality, and [launch] a collection of simulations around possible events," where there are no constraints on the number of input variables that can be used, and the analyst can ". . . generate thousands, perhaps millions, of random sample paths, and look at the prevalent characteristics of some of their features." [Taleb, 2004: 46]

Yet, the model is complex and involves a high degree of informed professional judgment throughout. For the simulation alone, the analyst must select those few variables that demonstrate significant influence over the resulting outputs, check for correlation among these, and ignore the rest. The analyst must also select the probability distribution and parameters for that distribution that represent a "best fit" for input variability. " . . . [B]ut, picking the right distribution and the parameters for the distribution remains difficult for two reasons. The first is that few inputs that we see in practice meet the stringent requirements that statistical distributions demand . . . The second is that the parameters still need to be estimated after the distribution is picked. . . . [yet the available data for this purpose is regularly insufficient or unreliable.]" [Damodaran, 2007: 165-167] The act of performing simulations may provide a mistaken sense of having rigorously investigated all aspects of a matter, when the adage "garbage in, garbage out," still prevails.

While this proposed methodology supplies a real-option like attempt to resolve what appear to be conflicting issues in more traditional methodologies and solve an important problem, it increases model risk by increasing model complexity and the need for subjective inputs without the true rigor of a real options approach.

Model 2: Valuation of complex tax issues related to organizational form: A debate has raged in the valuation community for years regarding the effect on value of the tax attributes belonging to Sub-Chapter S corporations. Should S-corporations be assigned higher values than C-corporations based on their tax attributes? After all, 1) investors in S-corporations avoid paying the dividend tax at the individual level but C-corporation investors cannot avoid this tax (Note that both investor classes pay taxes on income earned at the corporate level.); and 2) S-corporation shareholders can increase the tax basis in their stock through retained earnings, while C-corporation shareholders cannot. Even the Federal Tax Court has entered the debate, issuing a number of decisions since 1999 that have created further confusion and discussion.

To address this issue and more precisely quantify any additional value that S-corporation status might bring to its shareholders, five valuation experts have developed models. Four of these, each named for its progenitor, are complex enough to be virtually proprietary, although they have all made their way into practical use to one degree or another. The fifth method, the "Simplified Model," explains and compares the other four and offers a streamlined approach to modeling the same issues.

The fundamental components of the "Simplified Model" are: 1) A traditional discounted cash flow (which can be expanded for any holding period or contracted to a single period capitalization); 2) recognition of the benefit of the avoided dividend tax; and 3) recognition of the capital gains benefit of the ability to build up basis." [Fannon, 2007: 4-1] The model requires the analyst to consider, select, and quantify the following assumptions: 1) Annual distribution percentages and amounts; 2) the probability that the likely hypothetical buyer, under a fair market value standard, will quality to maintain the S-corporation status; 3) the level of risk associated with shareholder ability, or lack thereof, to realize basis build-up; 4) the estimated holding period before the hypothetical buyer will "flip" his investment in the company; and 5) the federal income tax rates to be used for the company and the shareholders. [Fannon, 2007: 4-1 through 4-2]

Each assumption requires the analyst to apply varying degrees of informed professional judgment based on varying sets of facts and analyst perspectives. Model complexity and risk are increased due to the number of factors to be considered and path dependencies that cannot be addressed. More importantly, these models may suggest that, if a company's tax attributes have a substantial influence on its value, should we not model the complete range of tax attributes of every company investigated during valuation analysis since companies have widely differing tax attributes based on their economic system design?

This would require an approach to modeling the economic system of the firm that is not currently available within traditional valuation practice.

A Simplified Model to Describe and Manage Complex Economic Systems

In *Strategic Investment: Real Options and Games,* Drs. Hans Smit and Lenos Trigeorgis synthesize corporate finance, industrial organization, corporate strategy (strategic planning), and value into a simplified model that describes and manages the complexity of the effect of firm optionalities and strategic games on value creation. Their basic premise is:

"In the past decade, the strategic management field has seen the development of two main views. One view is that flexibility is valuable. As the competitive environment of most firms changes quite frequently, flexibility in investments should allow firms to optimize their investments and value creation. The other view is that commitment is valuable because it can influence the strategic actions of competitors. This creates the opportunity to realize better payoffs (and shareholder value).

"Both views are supported by theoretical arguments and a large body of research. The flexibility view partly draws on the resource-based view of the firm and core-competency arguments: a firm should invest in resources and competencies that give it a distinctive chance to pursue a set of market opportunities. . . . The commitment view is firmly anchored in industrial organization and game theory, which during the nineties were increasingly adopted in the strategy field.

"Since both views have a theoretical justification, a key question is under what circumstances each can inform strategic decisions." [Smit & Trigeorgis: 2004, 35] Their response to their own question is:

Expanded (strategic) NPV = (passive) NPV + flexibility (option) value + strategic (game theoretic) value

Conceptually, they consider the firm as a portfolio of options, or ". . . 'bundle of opportunities' [requiring] a balance between exploiting current cash-generating advantages and generating new options." The correlations and interactions ("interproject synergies") and the "intertemporal (compound option) effects" among the firm's strategic and operational projects (options) as well as the risk attributes of various stages in these projects create the value of the firm (i.e. the portfolio of options). [Smit & Trigeorgis: 2004, 80-81] The value created by and within this portfolio can be quantified using a binomial option pricing model. This approach to valuing the firm's portfolio of options provides a richer and more realistic assessment of value than traditional, linear discounted cash flow models.

In addition, firms experience a "strategic impact" from the investment decisions they make in contexts in which they are aware of the actions and interactions of rivals that will affect project value. In such contexts, "[g]ame theory can be helpful in analyzing strategic investment decisions . . . [F]ollowing the rules of game theory can help reduce a complex strategic problem into a simple analytical structure consisting of four dimensions [(1) identification of the players, (2) the timing or order in which the players make their decisions, (3) the available actions and information set, and (4) the payoff structure attached to each possible outcome]. . . . [G]ame theory is also a helpful valuation tool for strategic decisions because it encompasses a solution concept that can help in understanding or predicting how competitors will behave, and it also provides an equilibrium strategy and values for the strategic decisions." [Smit & Trigeorgis: 2004, 171-172] Smit and Trigeorgis present an integrated model by which to discuss game theory in terms of real options analysis. This holistic model is simplistically described, following.

When a firm engages in multistage (sequential) games under uncertainty and wishes to analyze its strategic choices using game theory analysis, management will build a strategic decision tree by which to lay out available choices and moves. By moving backward through this decision tree structure, much like the certainty-equivalent binomial tree used in real options analysis, the option value at each node of the tree can be calculated using risk-neutral probabilities and options pricing methods. “This new approach makes it possible to value complete strategies in a competitive context in a fashion that is consistent with both modern economics and finance theory.” [Smit & Trigeorgis: 2004, 181] It also provides a simplified, but powerful, framework by which to investigate the effects of market competition and strategic planning on firm value. There are no traditional valuation tools that can address this important issue.

The same model might also be used in two other ways. First, it could be applied to overall firm design, rather than discrete project/strategy design. Second, it could be applied to the concept of *intra-organizational* competition within a sub-corporate finance context [Nelson, 2005]. Both applications would add new dimensions to our understanding of the firm and the real options “in” economic systems.

A Layered Model that Reduces Complexity during Use

“Real Options ‘in’ Projects and Systems Design – Identification of Options and Solutions for Path Dependency,” the Ph.D. dissertation by now Dr. Tao Wang, provides a rigorous discussion of the theoretical and computational aspects of real options “in” engineering systems. Wang’s dissertation has its origins in the seminal work of Dr. Richard de Neufville, his thesis supervisor, with whom he has published jointly a number of times. While, for practical reasons, this paper focuses on Wang’s dissertation, we should not overlook Dr. de Neufville’s overarching contribution to engineering systems design and real options “in” such systems.

Although Wang investigates engineering system design, we suggest that the proposed principals and methods can be extended to and provide powerful applications for the design and valuation of economic systems in general and the firm in particular. This section will provide a simplified overview of the proposed model in its original context. A later section will apply it to economic systems.

The design of physical systems, such as large-scale engineering systems, involves identifying and incorporating a vast array of technical constraints (real options "in" the system) that are highly interdependent and path dependent, conditions that are not traditionally considered in calculating the real options value "on" firm projects. An example of such path dependency is the manner in which the power generating capacity of a downstream dam changes when an upstream dam is built. [Wang, 2005: 20]

Traditional deterministic engineering systems design makes use of projected expected values of uncertain parameters, passive recognition of such uncertainties, and a focus on economies of scale. Dynamic engineering systems design of the sort proposed by Wang and de Neufville, considers "sequences of probability functions at multiple points in time," proactive management of the uncertainties, and foci other than economies of scale. [Wang, 2005: 22, 23] More provocatively, Wang also mentions the need to consider "social stochasticity," i.e. the economic and social consequences and uncertainties surrounding large-scale engineering project design, since "[a]ny technical systems are to serve human's needs." [Wang, 2005: 38]

The following describes the context in which dynamic engineering systems design has become a necessity: "Engineering systems are increasing in size, scope, and complexity as a result of globalization, new technological capabilities, rising consumer expectations, and increasing social requirements. Engineering systems present difficult design problems and require different problem solving frameworks than those of the traditional engineering science paradigm: in particular, a more integrative approach in which engineering system professionals view technological systems as part of a larger whole. Though engineering systems are very varied, they often display similar behaviors. New approaches, frameworks, and theories need to be developed to understand better engineering systems behavior and design." [Roos, 1998 from Wang, 2005: 29] Thus, engineering systems design and implementation require the consideration of high degrees of complexity and uncertainty, neither of which is properly captured using traditional, deterministic methodologies. Wang suggests a layered real options model that can address these challenges effectively.

First, the distinction between real options "on" systems (as traditionally computed) and real options "in" systems must be understood. Real options "on" systems "treat the technology as a black box," i.e. offer no consideration of or insight into the inner workings of the systems/projects they are valuing. [Wang, 2005: 106] Real options "in" systems consider the inner workings of system/project design to identify and provide flexibility (i.e., options) from the inside out.

The reason why real options "in" projects are of special interest to the study of engineering systems is that large-scale engineering projects share three major features that are particularly amenable to real options analysis. "They:

- Last a long time, which means they need to be designed with the demands of a distant future in mind;
- Often exhibit economies of scale, which motivates particularly large construction;
- Yet have highly uncertain future requirements, since forecasts of the distant future are typically wrong.

"This context defines the desirability of creating designs that can be easily adjusted over time to meet the actual needs as they develop." [Roos, 2004 in Wang, 2005: 107] Only some form of real options analysis could capture and quantify the flexibility required by such systems.

The top layer of Wang's approach is a screening model that answers the question: Which of the many options that present themselves in an engineering system are "most important and justify the resources for further study? The engineering system he uses as his test case is a dam building (i.e., water resources planning) project.

"The screening model is established to screen out the most important variables and interesting real options (flexibility). The screening model is a simplified, conceptual, low-fidelity model for the system. Without losing the most important issues, it can be easily run many times to explore an issue, while the full, complete high-fidelity model is hard to establish and costly to run many times. From another perspective . . . we can think of it as the first step of a process to reduce the design space of the system." [Wang, 2005: 138] The screening model involves simplifying assumptions such as allowing all sub-projects to be built at once and removing the stochasticity from all variables being explored (in this case, water flow and the price of electricity). If an important aspect of the project has been simplified in this manner, it should be studied in depth later, after the screening model is complete. In cases in which feedback exists in the system, the screening model must take it into account in order to ensure accurate results.

The screening model uses non-linear programming to perform sensitivity analysis on key system parameters. Once optimal designs have been identified for each set of parameters, they are reviewed and compared for real options that are both "good" for all sets and also conducive to optimal value creation.

The next layer in the overall model is a high-fidelity simulation model, through which the selected candidate designs are put. "Its main purpose is to examine, under technical and economic uncertainties, the robustness and reliability of the designs, as well as their expected benefits. . . . This process leads to refinement of the designs identified by the screening model."

[Wang, 2005: 140] Standard water resource planning simulations use historical records to simulate stochastic variation in water flows. Wang's model simulates both water flow and economic uncertainties in order to fully understand the role economies of scale should play, or not play, in the final design.

Once the most promising real options in the project design have been identified, these options must be valued so that a primary strategy and related contingent strategies may be set. Wang suggests "recasting [a standard binomial lattice] in the form of a stochastic mixed-integer programming model [in which the binomial tree is maximized] subject to constraints consisting of 0-1 integer variables representing the exercise of the options (=0 if not exercised, =1 if exercised." [Wang, 2005: 141, 142] "[S]uch reformulation empowers analysis of complex path-dependent real options 'in' projects for engineering systems.

"Technical constraints in the screening model are modified in the real options timing model. Since the screening and simulation models have identified the configuration of design parameters, these are no longer treated as design variables. On the other hand, the timing model relaxes the assumption of the screening model that the projects are built together all at once. It decides the possible sequence of the construction of each project in the most satisfactory designs for the actual evolution of the uncertain future." [Wang, 2005: 152]

To assist those readers who do not have an expertise in the kinds of programming used in this analysis, Wang provides the following descriptions: "Mathematical programming studies the mathematical properties of maximizing or minimizing problems, formulates real world problems using mathematical terms, develops and implements algorithms to solve the problems. Sometimes mathematical programming is mentioned as optimization or operations research. . . . Stochastic programming is the method for modeling optimization problems that involve uncertainty. . . . In stochastic programming, some data are random, whereas various parts of the problem can be modeled as linear, non-linear, or dynamic programming. . . . A mixed integer programming problem is the same as the linear or non-linear problem except that some of the variables are restricted to take integer values while other variables are continuous." [Wang, 2005: 39, 40, 41, 42] One deficiency of stochastic mixed-integer programming is that it is difficult to tell if the result is a global or a local optimum. However, Wang maintains that the solution provided by this computational methodology, while it may be a local rather than a global one, is superior to the solutions offered by traditional methodologies or human intuition. In addition, once the problem is programmed, it can be executed easily and rapidly on an ordinary laptop computer.

This layered approach to wringing the complexity out of and harvesting the value from engineering systems design suggests that, in spite of computational complexity, its application to economic systems should be further explored.

III. Love, Death, Taxes and Real Options "in" Economic Systems

Love, death and taxes – these are the inevitable, yet uncertain and risky, events of which life cycles and systems are composed. While we normally think of them in terms of human life cycles, they can be applied to economic life cycles as well, under different names. The description and quantification of love, death and taxes in human life are left to the social sciences, statisticians, actuaries, and poets. But the description and quantification of these in economic systems remain the task of economics, finance, and valuation. As economic systems become increasingly complex and dynamic and the universal language of historical accounting is being altered, the theory and tools we use in economics, finance, and valuation are beginning to prove inadequate to the tasks being required of them. Hence, there is a need to consider new avenues of thought and new tools. In this section, we explore the potential use of real options "in" systems design as a means to achieve more rigorous and insightful results in the design and valuation of the economic system of the firm.

What are economic systems? While there might be many informative responses to this question, the description of dynamic engineering systems provided by Wang earlier in this paper offers an interesting place to start. As with engineering systems, economic systems come in many sizes and forms. They can be as large as the global marketplace, a national economy, an industry, or a firm. They can be as small as a family unit, though no smaller since any system requires more than one agent to exist. Similar to other systems, economic systems are governed by explicit and implicit rules that affect all agents within the system in varying ways. The kinds and combinations of rules in a particular economic system allow it to be put into general system categories such "capitalism," "oligopoly," or "start-up." Economic systems should be open, complex, and adaptive - living. If they are not, they may be kept on life support but eventually they will die (as in the demise of the Iron Curtain) or rupture open in an unmanageable chaos of birthing (as with the hyper-capitalism being born out of that closed system, the People's Republic of China). Why? Because economic systems are created by and built upon open, complex, adaptive, living biological systems called human beings and life must beget life.

What is the dilemma this creates for those who study them? The very life-bearing properties of these systems, however, have created a dilemma. Until fairly recently, only limited means have been available to describe them quantitatively without the use of highly simplified, linear, deterministic models. While much of this is due to a general absence of powerful yet accessible technological tools, most of it is due to the fact that it is just plain more direct and less time-consuming to think about and use deterministic models. A broad-brush approach has been considered sufficient for most decisions and transactions. In addition, until fairly recently, those guardians of the national economic systems, the regulators, agreed.

Strangely enough, increasingly draconian oversight by the regulators may push theoreticians and practitioners in economics, finance and valuation to finally embrace approaches and methodologies that properly reflect the true nature of the economic systems they describe, measure, and manage. Real options analysis is one such approach.

The Firm as an Economic System

There are almost as many theories of the firm as there are researchers to build them. Such theories have been amply explored elsewhere and do not need re-examination. The goal here is to establish a solid foundation for the use of real options "in" the firm as an economic system. In order to accomplish this, we need to demonstrate: (1) that the inner life of the firm resembles a market in which risk and reward are critical determinants of value; and (2) that the firm is open, dynamic, complex, and adaptive like other systems.

Is the firm a market? The following description may help draw the parallel between the market endogenous to the firm and that exogenous to it. "Firm core processes are market participants, or agents, and are dynamically combining and recombining into portfolios of capabilities, while also competing for scarce resources (capital, knowledge assets, and infrastructure). Each core process is made up of a portfolio of sub processes (e.g., "supply chain") that enables it to carry out activities and produce outputs. Each core process is affected by the risk and uncertainty, the availability of resources, the property rights, the transaction costs, and the real investment options available to the firm and to other core processes within the firm.

" . . . [The] firm is more than the sum of its parts (processes), since these parts are constantly overlapping, competing, and shifting dynamically internally and externally as they interact with stakeholders and competitors. [In addition, the] firm and its agents (processes) are characterized by change, uncertainty, and risk. Differences between [exogenous markets and endogenous markets] with regards to change, uncertainty and risk are a matter of degree not kind. It is arguable as to which market experiences a higher degree of change, uncertainty and risk.

"The [firm] is created and maintained by various kinds of individual, institutional, and organizational investors and stakeholders. . . . Top management functions much like an investment fund or portfolio manager, deciding where to allocate scarce resources and add or divest investments that allow the firm to continue to function with varying degrees of health. Each core process has its own management team that also functions much like a portfolio manager." [Nelson, 2005: 39] These characteristics of the endogenous market of the firm are virtually identical to those of the exogenous market.

How do risk and reward factor into this "market" of the firm? There are many commonly discussed and important sources of risk and reward to the firm. Some are exogenous and some endogenous to it. All must be considered when discussing corporate strategy, considering corporate projects or transactions, attempting to identify the key value drivers of a firm or value the firm itself, invest in it or close it down. The challenging and ever changing nature of risk and its relationship to reward is perhaps the biggest source of discussion and debate in the boardroom, the halls of academia, the offices of practitioners, and on "The Street." Assessments of risk and reward are key components of value. Yet, as the economic system of the firm becomes increasingly complex and sophisticated and path dependencies between sources of risk and reward increase, traditional assessment approaches have less and less appeal.

While we can talk about risk and reward in the firm in relationship to its environment, we have not had any mechanism by which to discuss these issues as they relate entirely to the internal life system of the firm. The intention of this paper is not to discuss this problem in any detail, though it is a pressing one. However, a brief description of a proposed solution follows in order to provide a complete picture of the variables that might be addressed by the practice of real options "in" the economic system of the firm.

In 1998, Drs. Kanevsky (mathematics) and Housel (computer science) proposed a nugget of theory that opened the door to a different view of firm risk and reward. They suggested that it was possible to describe all organizational outputs in terms of common units of change. They stated, "Businesses are open systems – systems that exchange substance and energy with their environments. As such, businesses have the capability, through their processes, to change the structure of raw material inputs (i.e., substance, energy, information) into final products/services. In the language of thermodynamics, this change in structure can be measured in terms of the corresponding change in entropy, when the input state a is transformed into output state b by process P (i.e., $b = \mathrm{P}(a)$). Assume that this change can further be represented as a set of 'elementary' changes that are minute enough to become identical in terms of the corresponding amount of entropy they cause. This assumption about the equivalence of elementary changes can be expanded across any finite number of processes with predetermined outputs. This allows the comparison in terms of entropy among any set of processes by means of elementary changes. . . .

"This concept can be applied to calculating the value added by business processes by calculating the entropy of K-complexity caused by the process to transform an input to its process output. To accomplish this, we will employ the parallelism between business processes and computations. . . ." [Kanevsky and Housel, 1998: 278-80 from Nelson, 2005: 5-6]

"Since a unit of K-complexity represents a unit of change and is equivalent to a unit of Shannon information, all process outputs can be standardized by describing them in terms of the number of units of Shannon information (i.e., bits) required to produce them, given the state of the technology used in the process.

"All outputs can also be described in terms of the time required by an 'average' learner to learn how to produce them. 'Learning Time' can be considered a surrogate for the amount of organizational knowledge required to produce the outputs . . . [allows us to] describe outputs in terms of learning time [and to assign] a common unit, the Knowledge Unit (K_{μ}), . . . to represent the amount of organizational knowledge required to produce the outputs." [Nelson, 2005: 7]

Although the manner in which this theoretical base can be applied in practice is not pertinent to this paper, the proportionality between units of change, units of Kolmogorov Complexity, and units of information have profound implications for the concept of the firm as an economic system and the quantitative measurement of sub-corporate risk and reward. The following further develops the notion of sub-corporate risk and reward based on common units of change.

"*Entropy* is the term that describes the reduction of energy to a state of maximum disorder in which each individual movement (activity) is neutralized by statistical laws. Left to itself, an isolated system tends toward a state of maximum disorder, i.e. highest probability. Boltzmann stated, 'In an isolated system, the system will evolve to its most probable state, that is, the one with the most homogeneous probability distribution,' (e.g. the Law of Large Numbers). In a state of homogeneity (or, highest entropy or uncertainty), we have no indication at all to assume that one state is more probable than another.

"*Information* is a probabilistic measure of reduction in uncertainty (entropy). The following formula, developed by Claude Shannon, expresses the probabilistic relationship between entropy and information, for all possible states 1 . . . *n* :

$$\mathrm{H}(x) = -\sum_{i=1}^{n} p(i) \log_2 p(i)$$

Where:

H	=	Entropy
x	=	A discrete random event
p	=	Probability distribution
i	=	Outcome

"H is maximized if all states are equi-probable (a state of homogeneity), since when there is no pattern, there is no information and entropy and information are opposites. H is 0 if $p(i) = 1$, since the system is in a state of maximum certainty or complete information.

"Randomness, entropy, probability, and uncertainty are equivalent terms. Their opposites are pattern, complexity, information, and certainty which are also equivalent terms.

". . . In investing activities, we could call process *P* a 'transaction' that structures input *Asset A* into output *Asset B*. The structural change that occurs during a transaction process (*TP*) involves a change in uncertainty (entropy, Kolmogorov complexity) as *Asset A* undergoes a state transformation to become *Asset B*. The monetization of this change in uncertainty from *Asset A* to A*sset B* is what we commonly call 'return on investment.' It is also the 'value added' by *TP*.

"[Using this notion, we might consider the term] *risk* to be a descriptor for the *change in uncertainty* (ΔΦ) related to the state transformation of *Asset A* into *Asset B* via *TP*. As such, it is a rate and is composed of two elements: (1) *volatility*, the magnitude of change in uncertainty; and (2) *growth, or, drift*, the direction of change in uncertainty. The 'expected return' (i.e., expected $\Delta\Phi$) for *Asset A* remains the baseline against which to estimate the risk (i.e., actual $\Delta\Phi$) related to *Asset B* regarding the state transformation taking place via *TP*.

"Thus, when we adjust cash flows for *risk* we are providing an estimate of the $\Delta\Phi$ that will be assigned to those cash flows as they undergo a state transformation via *TP*.

"Since $\Delta\Phi$ is also a descriptor for 'return on investment' we now have a . . . way of describing the relationship between risk and return, e.g. they are equivalent. . . . Risk is actual $\Delta\Phi$ and return is expected $\Delta\Phi$, which enables us to use the traditional notion of matching actual risk with expected return.

"In addition, although risk is commonly described in the literature (Mun, 2002 and many others) as interchangeable with uncertainty, [this] approach indicates that *risk is no longer interchangeable with uncertainty*. Risk is the *change* in uncertainty ($\Delta\Phi$), not the uncertainty itself.

". . . If we apply the proportionalities we described [earlier] in which ΔE (change in entropy, uncertainty) $\approx K(y|x)$ (conditional Kolmogorov complexity) $\approx$ bits $\approx K_\mu$, and we agree that risk is a change in uncertainty ($\Delta\Phi$), then risk and K_μ are also proportionate and represent the same common unit of measure.

"As a result, we suggest that measuring the change in entropy embodied in process outputs of the organization in common units, K_μ, is equivalent to measuring risk. This in turn ties risk measurement *directly* to the knowledge assets of the organization and only indirectly to the movements of 'the market' and competitors." [Nelson, 2005: 28-29, 30, 31]

While this set of concepts is not currently in practice, it provides new avenues of thought and further rationale for considering the firm as a true economic system, complete with its own endogenous, quantifiable sources of risk and reward.

In what way is the firm a complex adaptive system rather than just a portfolio of resources, options, rights, knowledge assets, or projects? "The [firm] . . . exhibits all the attributes of a complex adaptive system. (1) It is a network of many agents acting in parallel, in an environment produced by the interactions of all agents on each other, and in which system control is highly dispersed. (2) It has many levels of organization, with agents at one level serving as building blocks for agents at a higher level, and in which the system is constantly revising and rearranging building blocks as it adapts and gains experience. (3) It anticipates the future by constantly making predictions based on its internal models of the world, its implicit and explicit assumptions of the way things are within the system. (4) It has many niches being exploited by agents adapted to fill each niche. The very act of filling a niche opens up more niches, creates new opportunities. Therefore, the [firm] is never in equilibrium because it is always unfolding, always in transition. [Waldrop, 1992:145-147]

"As an open, complex, adaptive system, the [firm] maintains a state of homeostasis – one of the most remarkable and typical properties of highly complex open systems. A homeostatic system maintains its structure and functions by means of a multiplicity of dynamic equilibriums rigorously controlled by interdependent regulation mechanisms. Such a system reacts to every change in the environment, or to every random disturbance, through a series of modifications of equal size and opposite direction to those that created the disturbance. The goal of these modifications is to maintain the internal balances. In a sense, the [firm] as an open system in homeostasis will be in a state of 100% information and risk ($\Delta\Phi$). This state corresponds to Dr. Stephen Wolfram's complexity theoretic universality class IV, the edge of chaos, the state in which '[o]rder and chaos intertwine in a complex, ever-changing dance of submicroscopic arms and fractal filaments' and life originates." [Waldrop, 1992:230, 231; Wolfram, 2002 from Nelson, 2005: 35-36]

Applications of Real Options "in" the Economic System of the Firm

Once we have established the firm as an economic system in its own right, complete with sources of risk and reward, we can begin to look at firm design and valuation from fresh perspectives. Just as engineering systems design and implementation require the consideration of high degrees of complexity and uncertainty, neither of which is properly captured using traditional, deterministic methodologies, so the design, implementation, and valuation of the economic system of the firm faces the same issues. The Wang layered real options model may also address these challenges to the firm effectively.

To illustrate this, consider the two examples of complex valuation problems discussed in Section Two.

Love: Complex capital structures arise from business acquisitions or in anticipation of business acquisitions. Both parties to the transaction hope that true love today will bloom into stunning riches tomorrow – while protecting themselves with "prenups" by designing safeguards into the structure of this future relationship. Although Wang's layered real options model is ideally suited for transaction design (pre-acquisition), we suggest it can also be considered for post-acquisition valuation purposes. The model would explore the real options available "in" the capital structure due to the need to optimize firm value under varying liquidity event scenarios and value them, including path dependencies, using stochastic mixed-integer programming.

Taxes: The complexity and range of the tax attributes of firms is enormous as firms seek to minimize the effect of taxes on corporate and investor wealth. As stated earlier, once we begin to take tax attributes into account for one aspect of the firm (here, its organizational structure), would it not be useful to also consider the effect of taxation on other value-creating or value-destroying aspects? Currently, the valuation of intangible assets under SFAS 141 includes quantification of the future tax benefits attributable to the amortization of such assets. In addition, when valuing privately held companies and using public companies as benchmarks, we are forced to ignore the tax attributes and tax rates of those public firms since they vary so widely and, in fact, may not be available to "outsiders." Yet, this is just the tip of the iceberg. If we look at the practice of financial accounting, much or most of its complex rules begin and end in issues of taxation. Take methods to record inventory or depreciate tangible assets as examples.

Wang's proposed layered model to explore taxation real options "in" the firm could provide assistance to financial managers and others who perform various aspects of organizational design, to executives during transactions, to valuation analysts who need to quantify tax effects and put them into financial statements. While considering the effects of taxation on every aspect of the firm seems both unnecessary and burdensome, using Wang's screening model, for instance, could narrow the design space for a particular firm and reduce the number of options for consideration under the simulation and real options models. Once the basic configuration of the model was established for a particular firm, it could be revisited and "tweaked" over time without having to start over again from the beginning.

Bigger fish: These are but two of the many complexities and value-creating-destroying issues that are investigated by valuation professionals on an hourly basis. Currently, we are forced to consider all of them sequentially. Yet, we know they actually represent interdependencies as well as path dependencies. What about the synergies among the various tangible and intangible resources, projects, core processes, or business units of the firm that we know are primary firm value drivers? What difference does it make to value if we look at the firm as a portfolio of resources, options, contractual rights, or projects? Theoretically, we could utilize Wang's methodology to consider several of these views simultaneously to see which one was a "best fit" to the specific facts and circumstances of a particular firm or to better understand how a firm might be a synergistic combination of all of them.

We suggest that, using Wang's proposed models to develop a rigorous and accessible tool for the practice of real options "in" the economic system of the firm, we could potentially address a much larger swath of the system design space simultaneously and provide richer, more firm-specific, useful valuations.

Looking even further out, if the practice of real options "in" the economic system of the firm proved fruitful, it might also prove fruitful for government as it considers and evaluates moves that affect a national economic system.

The Obstacle of Computational Complexity

Computational complexity is a very real obstacle to the further exploration and utilization of Wang's model for real options. "Stochastic Programming: Computational Issues and Challenges," attached to this paper as Exhibit A, discusses in this obstacle in some depth.

The question we must ask ourselves at this juncture is whether it is preferable to make increasingly complex, sequential, and subjective adjustments to various traditional model variables in order to attempt to capture firm complexity or to explore and build complex computational models that can be run on an average laptop computer and embed the ability to investigate the effects of all variables simultaneously. The former eventually leads us so deeply into a maze of informed professional judgment that we lose all sense of reality concerning the specific "firm as economic system" we are valuing. The latter, while requiring the use of informed professional judgment in structuring the rules by which the programming models are built, might allow us to explore large design/valuation spaces while minimizing the role of subjectivity and/or speculation.

Simple, deterministic computations built on convoluted estimates and opinions, or complex, dynamic computations built on simple rules and judgment calls? That may be the choice facing us in this matter.

IV. Where do we go from here and why does it matter?

Clearly, further research must be performed to understand if the Wang model can be extended in the manner proposed herein, within the time and resource constraints of normal firm and consulting environments. Then the challenges of building the models for the design space of an economic system must be addressed. In addition, it would be highly beneficial if consideration was given to means of applying the Smit-Trigeorgis model side by side the Wang model, since strategy and competitive games are major sources of value in firms. If these challenges could be successfully addressed and the resulting computational models made readily available to the finance and valuation communities, only our imaginations would be the limit to the further applications of these concepts.

Why does this matter? We return to the humdrum of financial accounting. Accounting regulators want accounting to function like finance, leading to the corruption of financial statements and market data and the effective disaggregation of the firm into a collection of resources and claims against them. But traditional finance and valuation do not currently have the tools or concepts to solve this problem. Both disciplines are being required to do what neither can.

We believe that the practice of real options "in" economic systems might be an answer.

Exhibit A

Stochastic Programming: Computational Issues and Challenges

From Encyclopedia of OR/MS, S. Gass and C. Harris (eds.)

Stochastic Programming: Computational Issues and Challenges

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Introduction

Stochastic programming deals with a class of optimization models and algorithms in which some of the data may be subject to significant uncertainty. Such models are appropriate when data evolve over time and decisions need to be made prior to observing the entire data stream. For instance, investment decisions in portfolio planning problems must be implemented before stock performance can be observed. Similarly, utilities must plan power generation before the demand for electricity is realized. Such inherent uncertainty is amplified by technological innovation and market forces. As an example, consider the electric power industry. Deregulation of the electric power market, and the possibility of personal electricity generators (e.g. gas turbines) are some of the causes of uncertainty in the industry. Under these circumstances it pays to develop models in which plans are evaluated against a variety of future scenarios that represent alternative outcomes of data. Such models yield plans that are better able to hedge against losses and catastrophic failures. Because of these properties, stochastic programming models have been developed for a variety of applications, including electric power generation (Murphy et al [1982]), financial planning (Cariño et al [1994]), telecommunications network planning (Sen et al [1994]), and supply chain management (Fisher et al [1997]), to mention a few. The widespread applicability of stochastic programming models has attracted considerable attention from the OR/MS community, resulting in several recent books (Kall and Wallace [1994], Birge and Louveaux [1997], Prékopa [1995]) and survey articles (Birge [1997], Sen and Higle [1999]). Nevertheless, stochastic programming models remain one of the more challenging optimization problems.

While stochastic programming grew out of the need to incorporate uncertainty in linear and other optimization models (Dantzig [1955], Beale [1955], Charnes and Cooper [1959]), it has close connections with other paradigms for decision making under uncertainty. For instance, decision analysis, dynamic programming and stochastic control, all address similar problems, and each is effective in certain domains. Decision analysis is usually restricted to problems in which discrete choices are evaluated in view of sequential observations of discrete random variables. One of the main strengths of the decision analytic approach is that it allows the decision maker to use very general preference functions in comparing alternative courses of action. Thus, both single and multiple objectives are incorporated in the decision analytic framework. Unfortunately, the need to enumerate all choices (decisions) as well as outcomes (of random variables) limits this approach to decision making problems in which only a few strategic alternatives are considered. These limitations are similar to methods based on dynamic programming, which also require finite action (decision) and state spaces. Under Markovian assumptions the dynamic programming approach can also be used to devise optimal (stationary) policies for infinite horizon problems of stochastic control (see also Neuro-Dynamic Programming by Bertsekas and Tsitsiklis [1996]). However, DP-based control remains wedded to Markovian Decision Problems, whereas path dependence is significant in a variety of emerging applications.

Stochastic programming (SP) provides a general framework to model path dependence of the stochastic process within an optimization model. Furthermore, it permits uncountably many states and actions, together with constraints, time-lags etc. One of the important distinctions that should be highlighted is that unlike DP, SP separates the model formulation activity from the solution algorithm. One advantage of this separation is that it is not necessary for SP models to all obey the same mathematical assumptions. This leads to a rich class of models for which a variety of algorithms can be developed. On the downside of the ledger, SP formulations can lead to very large scale problems, and methods based on approximation and decomposition become paramount. In this article, we will provide a road map for these methods, and point to fruitful research directions along the way.

Mathematical Models and Properties

Consider a model in which the design/decision associated with a system is specified via vector x_1. Under uncertainty, the system operates in an environment in which there are uncontrollable parameters which are modeled using random variables. Consequently, the performance of such a system can also be viewed as a random variable. Accordingly, SP models provide a framework in which designs (x_1) can be chosen to optimize some measure of the performance (random variable). It is therefore natural to consider measures such as the worst case performance, expectation and other moments of performance, or even the probability of attaining a predetermined performance goal. Furthermore, measures of performance must reflect the decision maker's attitudes towards risk. For example in the financial literature, it is common to model risk aversion through the use of a utility function.

The following mathematical model represents a general SP formulation in which the design/decision variable x_1 is restricted to the set X_1, and $\tilde{\omega}_1$ denotes a multi-dimensional random variable.

$$\operatorname*{Min}_{x_1 \in X_1} \quad f_1(x_1) + E[h_2(x_1, \tilde{\omega}_1)] \tag{1a}$$

$$\text{s.t.} \qquad P[g_1(x_1, \tilde{\omega}_1) \geq 0] \ \geq p_1 \tag{1b}$$

Here E denotes the expectation with respect to $\tilde{\omega}_1$ and P denote the probability distribution associated with $\tilde{\omega}_1$. The function g_1 is often modeled by a linear function and h_2 is the value function of another optimization problem as follows:

$$h_2(x_1, \omega_1) = \operatorname*{Min}_{x_2 \in X_2(x_1, \omega_1)} \quad f_2(x_2; x_1, \omega_1).$$

In the SP literature, the function h_2 is used to reflect costs associated with adapting to information revealed through an outcome ω_1. In financial applications, this function may reflect the utility associated with costs of rebalancing the portfolio. Because the function $E[h_2]$ is associated with a recourse action, it is referred to as the recourse function. Constraint (1b) is called a probabilistic (or chance) constraint. Such a constraint is used to model system reliability. We should mention that formulation (1) is somewhat more general than one usually finds in the SP literature. Historically, the probabilistic constraint (1b) is treated separately from models using the recourse functions (1a). However, including both types of functions within a model allows us to view the SP problems in a more cohesive manner.

While model (1) appears somewhat static, it is not difficult to glean a dynamic element in the formulation: note that the function h_2 is realized only after the design x_1 is in place. This sequential nature is an essential element of decision making under uncertainty. Indeed, if we define h_2 recursively, problem (1) may be looked upon as the first stage problem of a more extensive multistage formulation. To present the multistage generalization of (1), consider an N stage problem. Let the boundary conditions be given by $h_{N+1} = 0$ and let $\tilde{\omega}_0$ denote a degenerate random variable reflecting the deterministic information available prior to decisions in stage 1. For $t = 1, \ldots, N$, let ξ^t denotes the history prior to stage t (i.e $\xi^t = (\omega_0, \ldots, \omega_{t-1})$). Note that the decision variables in stage t depend on the history of the data process. Hence these variables are functions of random variables, and will be denoted $x_t(\xi^t)$. The entire history of decisions until stage t will then be represented as a superscripted vector $x^t(\xi^t) = (x_1(\xi^1), x_2(\xi^2), \ldots, x_t(\xi^t))$, or simply x^t. For $t = 2, \ldots, N$, we can now define the value functions

$$\begin{aligned} h_t(x^{t-1}, \xi^t) = \underset{x_t \in X_t(x^{t-1}, \xi^t)}{\text{Min}} \quad & f_t(x_t; x^{t-1}, \xi^t) + E[h_{t+1}(x^t, \tilde{\xi}^{t+1} | \xi^t)] \\ \text{s.t.} \quad & P[g_t(x^t, \tilde{\xi}^{t+1} | \xi^t) \geq 0] \ \geq p_t, \end{aligned}$$

where E denotes the conditional expectation and P denotes the conditional probability associated with the appropriate random variables. Using these functions in (1) yields a multistage SP formulation.

While we have used a DP-type recursion to state the SP problem, it is important to note that all random variables are path dependent, and furthermore, unlike DP, the statement of the problem does not constitute the algorithm. In fact, alternative statements of the multistage problem are also possible. Consider a formulation in which we allow the decisions to depend on the entire realization ξ^N. Let $x^N(\tilde{\xi}^N)$ denote a sequence of random vectors $(x_1(\tilde{\xi}^N), x_2(\tilde{\xi}^N), \ldots, x_N(\tilde{\xi}^N))$. It is important to note that such a policy cannot be implemented since decisions in stage t require the knowledge of the entire realization! Hence, the plans (denoted $x^N(\tilde{\xi}^N)$) cannot be feasible, unless, the decisions are such that x_t depends only on data available until stage $t-1$. As shown below we can incorporate such information constraints explicitly.

Let $\omega^t \equiv (\omega_t, \ldots, \omega_N)$. Since any outcome $\xi^N = (\xi^t, \omega^t)$ for any t, the decisions in stage t can be represented as a random vector denoted $x_t(\tilde{\omega}^t | \xi^t)$. Then the information constraints (also called the nonanticipativity constraints) may be stated as

$$x_t(\tilde{\omega}^t | \xi^t) - E[x_t(\tilde{\omega}^t | \xi^t)] = 0 \quad \text{almost surely.}$$

Since a non-separable objective function can be written as $E[f(x^N(\tilde{\xi}^N), \tilde{\xi}^N)]$, the inclusion of information (nonanticipativity) constraints provides a legitimate multistage model which does not appeal to either separability or recursion.

The formulations presented above impose very few restrictions. Perhaps the most important restriction imposed in a SP formulation arises from the assumption that randomness is exogenous and cannot be affected by decisions. In certain design problems, such an assumption may not be valid, and in these cases, the models outlined above are inadequate. Nevertheless, there is a large class of applications where randomness is exogenous (e.g. weather, loads, prices of financial instruments, market demands etc.), and SP models provide a sound approach.

The main challenge in designing algorithms for stochastic programming problems arises from the need to calculate conditional expectation and/or probability associated with multi-dimensional random variables. For all but the smallest of problems, we resort to approximations. The study of stochastic programming algorithms has therefore led to alternative ways of approximating problems, some of which obey certain asymptotic properties. This reliance on approximations has prompted researchers to study conditions for the convergence of approximations, and/or the convergence of solutions of approximate problems (to a solution of the original). Of course, conditions ensuring the former imply the latter, but the converse does not hold. Issues related to convergence of approximations can be addressed through the theory of epi-convergence (King and Wets [1991], Rockafellar and Wets [1998]) whereas issues pertaining to convergence of solutions of approximations (to a solution of the original) can be addressed through the notion of epigraphical nesting (Higle and Sen [1992], [1995]).

The computational challenges associated with SP problems vary a great deal with the class of problems being addressed. As with any large scale optimization problem, exploiting properties and the structure of problems provides the key to effective algorithms. We discuss properties associated with some important classes of SP problems, and then proceed to discuss the computational issues.

Some Properties of Stochastic Linear Programs with Recourse

For this class of problems, all functions and constraints are defined by linear/affine functions, and the probabilistic constraints are absent. This remains one of the more widely studied models, and most of the applications reported in the literature belong to this category (including the applications mentioned earlier). Problems of this type can be shown to be convex optimization problems, and the full power of convex analysis can be brought to bear on such problems. Notwithstanding such mathematical attractiveness, SLP problems lack one of the more desirable numerical properties, namely, smoothness. Only under very special circumstances (absolute continuity of random variables, Kall [1976]), can one expect (1a) to be differentiable.

Some Properties of Stochastic Mixed Integer Linear Programs

For this class of problems, we continue with the absence of probabilistic constraints. In a stochastic mixed integer linear program (S-MILP), if only the first stage decisions include integer restrictions, then the remaining problem inherits the properties of a SLP. This class of problems (with first stage integer variables) is similar to the problems originally envisioned by Benders in his seminal paper (Benders [1962]). In general though (i.e when integer variables appear in future stages) the S-MILP is much more challenging. For such problems, convexity of the objective function is far too much structure to expect. Indeed, the objective function (1a) can be discontinuous. However, by assuming that any setting of decision variables yields a finite objective value (i.e. complete recourse), and assuming a weak covariance condition (Schultz [1993]) the objective function can be shown to be lower semicontinuous.

Some Properties of Probabilistically Constrained Problems

These models are widely used to reflect grade of service constraints (e.g. Medova [1998]). The early work for this class of problems was restricted to normally distributed random variables. Prékopa [1971] showed that a much larger class of random variables yield the convexity property; he showed that if the function g (see (1b)) is linear/affine in x and randomness only appears additively, and the random variable has a log-concave probability density function, then the resulting feasible region is convex. However, for discrete random variables this is no longer true, and in this case, the set of feasible solutions can be represented as a disjunctive set (Sen [1993]).

Computational Issues and Challenges

The main computational challenges can be attributed to presence of multi-dimensional integration (to calculate either expectation or probability) within an optimization algorithm. Even in cases where the random variables are discrete, the total number of outcomes of a multi-dimensional random vector can be so large that calculations associated with the summations may be far too demanding. Hence even in the case of discrete random variables, one may have to resort to approximations. Discretizations and/or aggregations in multi-stage problems result in alternative data scenarios or sample paths. These scenarios may be organized in the form of a scenario tree which is a structure representing the evolution of information over the stages. In such a tree, two scenarios that share a common history until stage t are indistinguishable until that stage, and thereafter they are represented by distinct paths. Thus every distinct scenario represents a path from the root node to a leaf node of the scenario tree. In the absence of appropriate approximations, these trees can become extremely large, and the model difficult to manage and solve.

There are essentially two major approaches to generating approximations. One is based on aggregating data points, and another based on selecting data points. The former class of algorithms lead to successive approximation methods in which finer discretizations of the sample space are created based on the solution of an aggregated stochastic program. Methods based on data aggregation and successive refinements have been forwarded by several authors, and a survey for two stage problems can be found in Frauendorfer [1992]. More recently, Edirisinghe and Ziemba [1996] have reported solving two stage problems with approximately 20 random variables. For multistage problems, data-aggregation methods have been proposed in Frauendorfer [1994], but computational results are very limited.

The idea of selecting data points to create approximations arises mainly in the context of sample-based algorithms. If one uses a fixed sample, then it is necessary to perform a statistical analysis of the output, as suggested by the work of Romisch and Schultz [1991] and Shapiro [1991]. To obtain asymptotic results, Shapiro and Homem-de-Mello [1998] (see also sample path optimization, Robinson [1996]) suggest solving a sequence of sampled approximations, with increasing sample sizes. As the approximating problem becomes larger, each iteration may become substantially demanding. In order to speed up computations associated with such a method, it may be advantageous to update approximations generated in earlier iterations. One such method for two stage problems is the stochastic decomposition (SD) algorithm (Higle and Sen [1991]) which incorporates sampling within a decomposition method. This combination allows the SD method to update approximations from one iteration to the next, thus allowing matrix updates and warm starts during re-optimization. Because sampling and decomposition are intimately interwoven in the SD algorithm, it allows the possibility of using empirical data directly within the algorithmic process. A detailed exposition of this work appears in Higle and Sen [1996], and recent results are summarized in Higle and Sen [1999]. As with methods based on data-aggregation, computational results with sample-based algorithms for multistage problems are extremely limited.

Most of the approximation schemes mentioned above are paired with some deterministic algorithm. This genre of methods traces back to the L-shaped method of Van Slyke and Wets [1969] which builds on arguments similar to Benders' decomposition (Benders [1962]) for two stage problems. The method has been extended in several ways, including generalizations to multistage problems (Birge [1985]). When the number of scenarios is small, these methods can be applied directly. Otherwise, they should be used in conjunction with approximation-based methods such as those discussed above.

Another class of deterministic decomposition algorithms is based on relaxing the information (nonanticipativity) constraints. This approach is particularly promising for parallelizing algorithms for multistage problems. Two such methods are the progressive hedging method of Rockafellar and Wets [1991] and diagonal quadratic approximation method of Mulvey and Ruszczyński [1995]. One of the biggest advantages of these methods is that they retain the structure of a deterministic counterpart (e.g. network structure) and are easily parallelizable. Furthermore, each processor can be allocated a collection of scenarios which can be coordinated with minimal oversight. Nielsen and Zenios [1993] report significant speed-ups of their parallel implementation over a serial code. It would be interesting to design sample-based algorithms of this type, and although some preliminary steps have been taken, it is too early to tell how such methods will perform.

One of the more demanding problems in stochastic programming involves the solution of stochastic mixed integer linear programs (S-MILP). In cases where the first stage has binary variables, Laporte and Louveaux [1993] have proposed an extension of the L-shaped method for two stage S-MILP problems. Unfortunately, it requires that the second stage problem (possibly a mixed-integer linear program (MILP)) be solved to optimality. Given the computational difficulties associated with MILPs, this is cumbersome. One possible way to alleviate this difficulty may involve an extension that incorporates the stochastic branch and bound algorithm proposed by Norkin, Ermoliev and Ruszczyński [1998]. Since the latter allows the use of sampled bounds, it may provide a more computationally feasible approach to practical S-MILP problems.

Finally, one area that has not attracted as much attention as it should is the development of software systems which integrate stochastic modeling with stochastic programming algorithms. Such a system would provide software tools to build models, validate them, and experiment with alternative algorithms. With few exceptions (e.g. Kall and Mayer [1996] and Gassmann and Ireland [1996]), there has been relatively little activity in this important area. It is unlikely that stochastic programming will attain its potential without the development of systems which allow easy interactions between stochastic models and stochastic programming algorithms. The development of a high level language or system that allows manipulation and representation of models and data, together with the ability to experiment with alternative solvers, is long overdue.

Before closing this article, we should point the reader to an extensive list of papers maintained by Maarten van der Vlerk at the following web site.

`http://mally.eco.rug.nl/biblio/SPlist.html`

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